21-378 **Mathematics of Fixed Income Markets** **Fall 2016**

Midterm Exam

90 Minutes; Closed-Book; Closed-Notes; Calculators are Permitted; No Laptops, Tablets or Cell Phones: **120 Points Total** Today's date is $t = 0$. All coupon bonds make payments every 6 months, with the first payment to be made 6 months from today. All interest rates and yields to maturity are expressed according to the semiannual compounding convention. All interest rates and yields should be assumed to be strictly positive. The coupon rates for coupon bonds are strictly positive. Some problems may contain information that is not needed for solving the problem.

Important: If you cannot solve a problem as stated, but you can solve it by making a small change in the problem statement, go ahead and solve the modified problem making clear that you realize that you have modified the problem. Depending on how significant your modification is, you may receive substantial partial credit. **You do NOT need to solve the problems in order.** If you do not see how to solve a problem immediately, skip it for the time being and come back to it later. If you get an answer that does not make sense and you cannot find a computational error, please indicate that the answer does not make sense and briefly explain why not.

1. (12 pts) Suppose that zero-coupon bonds of maturities 6 months, one year, and 18 months are trading at the prices (per \$100 face) shown in the table below.

Maturity	**Price**
.5 years	95.578
1 year	92.271
1.5 years	88.892

   (a) Find the spot rate $\hat{r}(1)$ and the forward rate $f(1)$.

   (b) A client walks into a bank today and agrees to borrow \$100,000 at time .5 and repay the loan with a single payment of amount A at time 1.5. Nothing is paid by either party to enter into the agreement. Determine A.

2. (16 pts) Assume that the annuity yield for maturity 15 years is $y_a(15) = .048$. Assume further that the discount factor for 12 years is $d(12) = .5529$ and that

$$r^{for}_{0,12,15} = .053.$$

   Here, $r^{for}_{0,12,15}$ is the forward rate agreed upon at time 0 to borrow or invest between $t = 12$ and $t = 15$ computed with semiannual compounding. Nothing is paid at time 0 to enter a forward loan agreement. Find $y_{pc}(15)$, the par-coupon yield for maturity 15 years.

3. (24 points) Suppose that the spot rate curve is flat at some level $y > 0$. Consider the following securities:

- Security #1 is a par-coupon bond with face value \$200,000 and maturity 15 years.
- Security #2 is a coupon bond with face value \$100,000, maturity 15 years, and coupon rate q with $y < q < 2y$.
- Security #3 is an annuity with maturity 15 years that pays $\$100,000\frac{q}{2}$ at each of the times $t = .5, 1, 1.5 \cdots, 15$, where q is the coupon rate for Security #2.

(a) Order the securities by (Macaulay) duration form largest to smallest if possible, or explain why such an ordering is not possible based on the information given.

(b) Order the securities by DV01 from largest to smallest if possible, or explain why such an ordering is not possible based on the information given.

4. (24 pts) Assume that the spot-rate curve is flat at .08. Portfolio A holds two zero-coupon bonds: one with face value $F_1 = \$5,000,000$ and maturity $T_1 = 7$ years and one with face value $F_2 = \$1,000,000$ and maturity $T_2 = 20$ years.

   (a) Find the Macaulay duration and DV01 for portfolio A. You may take it for granted that the current prices of these bonds are $P_1 = \$2,887,375$ and $P_4 = \$208,290$

   (b) Use a first-order approximation to find the approximate price change in Portfolio A resulting from a parallel shift upward of 24 basis points in the spot rate curve. (Be sure to specify both the direction and magnitude of the price change.)

   (c) Portfolio B is to consist of one par-coupon bond having face value F_3 and maturity $T_3 = 10$ years. Find a value forF_3 such that Portfolios A and B have the same DV01

5. (20 pts) Consider a 15-year non-standard interest-rate swap in which the notional principle doubles one third of the way through the swap and then returns to the original value two thirds of the way through the swap. More precisely, at each of the times $t = .5, 1, 1.5, 2, \cdots, 5$

   A pays B the variable amount $\frac{F}{2}r_{t-.5,t}$, and

   B pays A the constant amount $\frac{F}{2}q$;

   at each of the times $t = 5.5, 6, 6.5, \cdots, 10$

   A pays B the variable amount $Fr_{t-.5,t}$, and

   B pays A the constant amount Fq;

   and at each of the times $t = 10.5, 11, 11.5, \cdots, 15$

   A pays B the variable amount $\frac{F}{2}r_{t-.5,t}$, and

   B pays A the constant amount $\frac{F}{2}q$.

The swap rate q is chosen so that neither party pays anything to enter into the agreement. (The notional principal F is not ever paid.) Here $r_{t-.5,t}$ is the spot-rate that will prevail at time $t - .5$ for loans initiated at time $t - .5$ and settled with a single payment at time t. Determine q in terms of the discount factors $d(.5), d(1), d(1.5), \cdots d(15)$. Be sure to explain your reasoning fully and carefully.

6. (12 pts) Assume that the spot-rate curve is upward sloping, i.e. $\hat{r}(T+.5) > \hat{r}(T)$ for all $T = .5, 1, 1.5, \cdots, 29, 29.5$. At time 0, you enter a standard 30-year interest rate swap in which you will be receiving fixed and paying floating. (The fixed rate is chosen so that neither party pays anything to enter the swap agreement.) Let V_1 denote the value of your position on the swap at time 1, ignoring the payments made at times .5 and 1. (In other words, V_1 is the value at time 1 to receive fixed at the original swap rate and pay floating at teach of the times $1.5, 2, 2.5, \cdots, 29.5, 30$ on the original notional principal.))

   Suppose that the term structure at time 1 is the same as the original term structure, i.e.

   $$r_{1,T+1} = \hat{r}(T) \text{ for all } T = .5, 1, 1.5 \cdots, 29.5, 30.$$

   (Here, $r_{1,T+1}$ is the spot rate that will prevail at time 1 for borrowing between time 1 and time $T + 1$, expressed with semiannual compounding.) Based on this information, is it possible to determine whether $V_1 > 0$, $V_1 = 0$, or $V_1 < 0$? If so, explain which of the three alternatives must occur. If not, explain why not.

7. (12 pts) Assume that the spot rate curve is flat at 4%. Some important economic news will be announced tomorrow and this news will have an impact on spot rates. You are not sure whether rates will move up or down, but you are convinced that after the news is announced the 5-year spot rate will be higher than the 2-year spot rate. Let $\hat{r}_{new}(2)$ and $\hat{r}_{new}(5)$ denote the 2- and 5-year spot rates after the announcement. (Do not worry about the fact that a bond issued today with maturity T years will have maturity slightly less than T years tomorrow.)

   Suggest a trade involving 10,000,000 face of the 5 year ZCB and some amount of face of the 2-year ZCB that will make money if $\hat{r}_{new}(5)$ and $\hat{r}_{new}(2)$ shift small amounts away from 4% and $\hat{r}_{new}(5) > \hat{r}_{new}(2)$. Be sure to specify the face amount of the 2-year ZCB and specify for each of the two bonds whether your position will be long or short. (Use a first-order approximation.)